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AN INTEGRAL BOUND ON THE STRAIN ENERGY FOR THE TRACTION PROBLEM IN NONLINEAR ELASTICITY WITH SMALL STRAINS

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AN INTEGRAL BOUND ON THE STRAIN ENERGY FOR THE TRACTION PROBLEM IN NONLINEAR ELASTICITY WITH SMALL STRAINS

Joseph J. Roseman*

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ABSTRACT

For the traction boundary value problem in nonlinear elastostatics for a body which is convex in its undeformed reference state and with the assumption of sufficiently small strains (but not necessarily small displacement gradients), an upper bound is obtained for the elastic strain energy in terms of the L_2-integral norms of the surface tractions and body forces with the constant depending only upon the ratio of the outer and inner diameters and the physical constants of the material.

This result extends previous known results in linear elasticity (infinitesimal displacement gradients) and finite elasticity (small but finite displacement gradients) into the small strain theory of nonlinear elasticity.

AMS (MOS) Subject Classifications: 73C50, 73G05

Key Words: nonlinear elastostatics, strain energy bound

Work Unit Number 2 (Physical Mathematics)

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SIGNIFICANCE AND EXPLANATION

Suppose that an elastic body is in a natural stress-free state and is then deformed by the action of applied surface tractions and internal body forces. This situation is described mathematically by a set of highly complex nonlinear partial differential equations.

In order to simplify the equations, it is often assumed that, with the possible exception of a rigid body motion, each point in the undeformed body has been displaced only an infinitesimal amount (linear elasticity) or, more generally, by a finite but small amount (finite elasticity).

However, there are many situations in which points in the body undergo relatively large displacements, all of which can not be included in a single rigid motion of the body, and yet the strains produced are relatively small. This is true for example in the case of long thin bodies.

Therefore, it is desirable to obtain results using only the assumption that the strains are small (but finite) while the displacements and displacement gradients are relatively large. In the present research, an upper bound for the energy in the body in terms of the surface tractions and body forces is obtained using only the small strain assumption whereas previous similar results had been obtained under the assumptions of linear or finite elasticity. This result is therefore applicable in many cases where the previous results were not.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC and not with the author of this report.

AN INTEGRAL BOUND ON THE STRAIN ENERGY FOR THE TRACTION PROBLEM IN NONLINEAR ELASTICITY WITH SMALL STRAINS

Joseph J. Roseman*

1. INTRODUCTION.

An elastic isotropic and homogeneous body is assumed to possess a natural stress-free state in which it occupies a convex domain $\mathcal{D}_R \subset E^3$. It is then subjected to the combined action of applied surface tractions and internal body forces and arrives at a new equilibrium state in a domain $\mathcal{D}$. Using standard equations of nonlinear elastostatics and working in the context of small strain theory, we shall derive an estimate for the strain energy in the deformed body in terms of the L_2 integral norms of the surface tractions and the body forces with the constants involved depending upon the material properties and upon the geometry of the domain. The estimate is similar to one previously obtained by Bramble and Payne [1] in the context of linear elasticity and by Breuer and Roseman [2] in the context of finite elasticity.

The theory of mathematical elastostatics (cf. [3], [4]) assumes that the deformation from $\mathcal{D}_R$ to $\mathcal{D}$ can be described mathematically as a smooth one to one mapping from the domain $\mathcal{D}_R$ onto the domain $\mathcal{D}$ and that there is a relation between the geometric strain associated with the mapping and the stress in the deformed state. The fact that the deformed body is in equilibrium imposes conditions on the stress tensor which in turn leads to a set of second order nonlinear elliptic partial differential equations for the mapping vector function. These equations, together with the known conditions at the boundary, constitute an elliptic boundary value problem.

The equations of classical linear elastostatics are obtained by a linearization of this nonlinear system, a linearization which is, in effect, equivalent to treating

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the derivatives of the displacement vector (the displacement being the difference between the position of a point in $\mathcal{D}$ and its former position in $\mathcal{D}_R$) as infinitesimally small quantities. In theory, therefore, the equations of linear elasticity describe the physical model only for infinitesimally small motions, although experience has shown that they give good results in many applications.

In the theory of finite elasticity, it is assumed that the displacement gradients are sufficiently small, depending upon the geometry of the body and the material constants, but not infinitesimal. Finite elasticity should therefore provide a more accurate mathematical description of the physical model than linear elasticity and results in finite elasticity theory concerning existence, uniqueness and continuous dependence on the given data would, at the least, be valid wherever linear elasticity results are valid.

However, as Fritz John [5] points out, in practice it is possible to have small strains and relatively large displacement gradients, especially in thin rods or shells (indicating large relative "rotation" of different parts of the body). Thus, it would be especially meaningful to obtain results based on the nonlinear equations assuming only that the strains are sufficiently small (but not infinitesimal), while the displacement gradients are bounded but are permitted to be large in comparison with the strains. That is what is done in this paper.

The techniques employed here are a refinement of those used in [2] and can be extended in a straightforward manner to non-isotropic and non-homogeneous bodies. The convexity requirement on $\mathcal{D}_R$ is imposed in order to ensure the applicability of some work of F. John [6], [7] on the relationship between rotation and strain and can be relaxed somewhat.

Other results concerning bounds for the strain energy in elastostatics in terms of the given data include [1], [8], [9], [10] in linear elasticity, [2], [11], [12] in finite elasticity, and [13] for the displacement boundary value problem in nonlinear elasticity with small strains.* A description of some of the above mentioned results is given in [14].

*In both [13] and [14], the results in [13] are described as being within the framework of finite elasticity. However, the analysis in [13] actually requires only an a priori assumption of small strains within the elastic body.

2. PRELIMINARIES.

In the absence of surface tractions or internal body forces, an isotropic, homogeneous elastic body occupies a convex domain $\mathcal{D}_R \subset E^3$ with inner and outer diameters d and D respectively and with a boundary, $\partial\mathcal{D}_R$, which is at least smooth enough for the application of the divergence theorem. Under the action of a system of known body forces and surface tractions, $\mathcal{D}_R$ is mapped onto a domain $\mathcal{D}$ with boundary $\partial\mathcal{D}$ with a point (x_1,x_2,x_3) in $\mathcal{D}_R$ being taken to the point (y_1,y_2,y_3) in $\mathcal{D}$. After deformation, the body is in static equilibrium and the mapping from $\mathcal{D}_R$ to $\mathcal{D}$ is assumed to be one to one and sufficiently smooth.

The displacement vector u_i is defined as

$$u_i = y_i - x_i \tag{2.1}$$

and, since the mapping is one to one, u_i may be regarded as a function of either (x_1,x_2,x_3) or (y_1,y_2,y_3).

We next define*

$$p_{ik} = \frac{\partial y_i}{\partial x_k} = \delta_{ik} + \frac{\partial u_i}{\partial x_k}, \tag{2.2}$$

$$p^{ik} = \frac{\partial x_i}{\partial y_k}, \tag{2.3}$$

$$g_{ik} = p_{ji}p_{jk} = g_{ki}, \tag{2.4}$$

$$g^{ik} = p^{ij}p^{kj} = g^{ki}. \tag{2.5}$$

The tensors p_{ik} and p^{ik} are, respectively, the covariant and contravariant tensors, g_{ik} and g^{ik} are the covariant and contravariant metric tensors, and δ_{ik} is the Kronecker delta (identity tensor).

---

*Tensor notation is used throughout the paper and all indices may take on the values 1, 2 and 3. The summation convention is followed so that a repeated index in any term is summed over all values of the index. The magnitudes of a (real) vector v_i and matrix b_{ik} are defined here as $|v| = \sqrt{v_i v_i}$ and $|b| = \sqrt{\frac{1}{3} b_{ik}b_{ik}}$.

We have the relations

$$p^{ij}p_{jk} = p_{ij}p^{jk} = \delta_{ik} \tag{2.6}$$

and

$$g^{ij}g_{jk} = g_{ij}g^{jk} = \delta_{ik} \; . \tag{2.7}$$

There are several definitions in the literature for the strain tensor in nonlinear elasticity. One of the more common, which we shall use here, is

$$e_{ik} = \frac{1}{2}\,(g_{ik} - \delta_{ik}) \; . \tag{2.8}$$

From (2.2) and (2.4), this is equivalent to

$$e_{ik} = \frac{1}{2}\,\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_k}\right) \; , \tag{2.9}$$

or, in terms of y_i coordinates,

$$e_{ik} = \frac{1}{2}\,\left(\frac{\partial u_i}{\partial y_k} + \frac{\partial u_k}{\partial y_i} + p_{mk}\frac{\partial u_j}{\partial y_m}\frac{\partial u_i}{\partial y_j} + p_{mi}\frac{\partial u_j}{\partial y_m}\frac{\partial u_k}{\partial y_j} + p_{ji}p_{mk}\frac{\partial u_n}{\partial y_j}\frac{\partial u_n}{\partial y_m}\right) \; . \tag{2.10}$$

The medium is assumed to be hyperelastic, i.e. it possesses a positive definite strain energy density function $W = W(e)$ which has the properties that for every $G \subset \mathcal{D}$ whose undeformed preimage is $G_R \subset \mathcal{D}_R$, the strain energy in G, $U(G)$, is given by

$$U(G) = \iiint\limits_{G_R} W(e)\,dx_1 dx_2 dx_3 \tag{2.11}$$

and

$$W(e) = \mu e_{ik}e_{ik} + \frac{\lambda}{2}\,e_{ii}e_{kk} + O(e^3) \; , \tag{2.12}$$

where $O(e^3)$ is a smooth term of order of magnitude $|e|^3$ with a derivative of order $|e|^2$ for small $|e|$ and λ and μ are positive material constants.

In addition, there exist material constants m and M such that

$$m|e|^2 \leq W(e) \leq M|e|^2 \tag{2.13}$$

if $|e|$ is sufficiently small.

The stress at any point in $\mathcal{D}$ is represented by the Cauchy stress tensor τ_{ik} which has the property that for any differential surface element dS in $\mathcal{D}$ with unit normal vector ν_k, the components of force on this element are $\tau_{ik}\nu_k dS$.

The body in the deformed state is in equilibrium under the action of the body forces per unit volume F_i and surface tractions (force/area) T_i.

The tensor τ_{ik} satisfies the equations

$$\frac{\partial \tau_{ik}}{\partial y_k} = -F_i, \qquad y_i \in \mathcal{D} , \tag{2.14a}$$

and the boundary conditions

$$\tau_{ik} n_k = T_i, \qquad y_i \in \partial\mathcal{D} , \tag{2.14b}$$

where n_i is the outward unit normal to $\partial\mathcal{D}$.

The existence of the strain energy density function $W(e)$ implies the following relation between τ_{ik} and e_{ik}:

$$\tau_{ik} = \frac{1}{\det(p)} p_{km} \frac{\partial W(e)}{\partial p_{im}} , \tag{2.15a}$$

(cf. Stoker [3, pp. 10-14]) or, from (2.4) and (2.8),

$$\tau_{ik} = \frac{1}{\det(p)} p_{in} p_{km} \frac{\partial W(e)}{\partial e_{nm}} . \tag{2.15b}$$

Equation (2.15b) shows that the tensor τ_{ik} is symmetric (because e_{ik} is symmetric).

For the problem considered here, it is assumed a priori that, throughout $\mathcal{D}$, $|e|$ is sufficiently small with respect to the geometry of $\mathcal{D}_R$ and the material constants; more precisely we assume that

$$|e| \leq \delta\left(\frac{d}{D}\right)^3 \tag{2.16}$$

with δ sufficiently small depending upon the material.

However, the only restrictions that we require on the displacement gradients are that the mapping from $\mathcal{D}_R$ to $\mathcal{D}$ be one-to-one, that the magnitudes of the matrices

p_{ik}, p^{ik}, g_{ik} and g^{ik} and their determinants be less than or equal to two in absolute value, and that $\mathcal{D}$ satisfy some mild geometric requirements, which are stated in the theorem below.

3. THE ENERGY BOUND.

We now state the main theorem: Consider an elastic, isotropic body which possesses a stress-free reference state in which it is homogeneous and occupies a convex domain $\mathcal{D}_R \subset E^3$ with inner and outer diameters d and D respectively. Suppose that the body is subjected to internal and external forces and reaches a state of static equilibrium in which it occupies a domain $\mathcal{D} \subset E^3$ with internal body forces F_i (force/unit volume) throughout $\mathcal{D}$ and surface tractions T_i (force/unit area) on $\partial\mathcal{D}$, the boundary of $\mathcal{D}$.

Assume that each point (x_1,x_2,x_3) in $\mathcal{D}_R$ is mapped to a point (y_1,y_2,y_3) in $\mathcal{D}$ in a smooth, one to one manner, that the magnitudes of the covariant and contravariant Jacobian and metric matrices and the determinants of these matrices are bounded in absolute value at every point by two, and that the strain matrix satisfies (2.16).

Suppose that $\partial\mathcal{D}$ has continuous curvature with maximum principal curvature not greater than 32/d and that at every point on $\partial\mathcal{D}$ there exists a sphere of radius d/64 which is tangent to the surface at the point and whose interior lies entirely within $\mathcal{D}$. In addition, there exists a point in $\mathcal{D}$, which is taken to be the origin, with the properties that the interior of the sphere $\Sigma : y_i y_i = \frac{d^2}{16}$ is contained in $\mathcal{D}$ and $y_i n_i / \sqrt{y_k y_k} \geq -\frac{1}{4}$ for all (y_1,y_2,y_3) on $\partial\mathcal{D}$, where n_i is the outward unit normal vector.

Assume that there exists a strain energy density function for the material which satisfies the relations (2.12) and (2.13).

Then, if δ (Equation (2.16)) is sufficiently small with respect to the material constants, the total elastic energy of the body $U(\mathcal{D})$ satisfies the inequality

$$U(\mathcal{D}) \leq BdD\left(\frac{D}{d}\right)^q \left[\|F\|^2 + \frac{1}{D}\|T\|^2\right], \qquad (3.1)$$

where

i) $\|F\|$ is the L_2 integral norm of $|F|$ over $\mathcal{D}$,

ii) $\|T\|$ is the L_2 integral norm of $|T|$ over $\partial\mathcal{D}$,

iii) q is a sufficiently large positive universal constant, and

iv) B is a constant which depends only upon the physical constants of the material.

The rest of Section 3 consists of the proof of this result. Throughout the following, all constants denoted by K_i are universal, while those denoted by C_i may depend also upon the material constants.

We begin by noting that, because of (2.12), (2.13) and (2.15), and because of the fact that the magnitudes and determinants of p, p^{-1}, g, and g^{-1} are uniformly bounded, that the integrals over either $\mathcal{D}_R$ or $\mathcal{D}$ of the expressions $W(e)$, $e_{ik}e_{ik}$, $\tau_{ik}\tau_{ik}$, and $\tau_{ik}e_{ik}$ are all of the same order of magnitude in the sense that any one is bounded in terms of any other with the constant depending only upon the material constants. We also see that

$$|\tau| \leq C_1 \delta \left(\frac{d}{D}\right)^3 . \tag{3.2}$$

We now call attention to the fact that if the deformed body were now subjected to any rigid body motion (that is, any combination of translations and rotations), the magnitude of the strain tensor e at any point as it moves with the body would remain invariant and so would the magnitude of the stress tensor τ, the strain energy $U(\mathcal{D})$, and the integral norms of the body forces and the surface tractions (cf. [15, pp. 127-128]). Therefore, since all the quantities of interest in (3.1) are unaffected, we shall assume that a rigid motion has already taken place and has been included in the mapping from $\mathcal{D}$ to $\mathcal{D}_R$. The rigid motion has been chosen so that the body is in what we consider to be a convenient orientation, as will be explained below.

The papers of Fritz John [6], [7], on the relationship between rotation and strain imply that if the quantity δ in (2.16) is sufficiently small, the body may be rigidly rotated to a position in which an inequality of the following type is valid:

$$\iiint_{\mathcal{D}_R} \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}\, dx_1 dx_2 dx_3 \leq K_1 \frac{D^3}{d^3} \iiint_{\mathcal{D}_R} e_{ik}e_{ik}\, dx_1 dx_2 dx_3 . \tag{3.3}$$

Such a position is one to which we have referred as convenient in the paragraph above.

We note that the integrand on the right side of (3.3) is invariant under all translations and rotations while the integrand on the left depends upon the orientation of the body.

Because of the bounds on p and p^{-1}, we also have

$$\iiint_{\mathcal{D}} \frac{\partial u_i}{\partial y_k} \frac{\partial u_i}{\partial y_k} dy_1 dy_2 dy_3 \le K_2 \frac{D^3}{d^3} \iiint_{\mathcal{D}} e_{ik} e_{ik} dy_1 dy_2 dy_3, \tag{3.4}$$

and, from the remarks in the paragraph preceding (3.2),

$$\iiint_{\mathcal{D}} \frac{\partial u_i}{\partial y_k} \frac{\partial u_i}{\partial y_k} dy_1 dy_2 dy_3 \le K_3 \frac{D^3}{d^3} \iiint_{\mathcal{D}} \tau_{ik} e_{ik} dy_1 dy_2 dy_3 . \tag{3.5}$$

From this point on, we shall use dV_x, dS_x, dV_y and dS_y to denote differential volume and surface elements in $\mathcal{D}_R$ and $\mathcal{D}$.

From (2.10) and from the fact that τ_{ik} is a symmetric tensor, it follows that

$$\begin{aligned} \iiint_{\mathcal{D}} \tau_{ik} e_{ik} dV_y = \iiint_{\mathcal{D}} \tau_{ik} \frac{\partial u_i}{\partial y_k} dV_y \\ + \iiint_{\mathcal{D}} p_{mk} \tau_{ik} \frac{\partial u_j}{\partial y_m} \frac{\partial u_i}{\partial y_j} dV_y \\ + \frac{1}{2} \iiint_{\mathcal{D}} p_{ji} p_{mk} \tau_{ik} \frac{\partial u_n}{\partial y_j} \frac{\partial u_n}{\partial y_m} dV_y . \end{aligned} \tag{3.6}$$

Using the divergence theorem and noting (2.14), we obtain

$$\begin{aligned} \iiint_{\mathcal{D}} \tau_{ik} e_{ik} dV_y = \iiint_{\mathcal{D}} F_i u_i dV_y + \iint_{\partial\mathcal{D}} T_i u_i dS_y \\ + \iiint_{\mathcal{D}} p_{mk} \tau_{ik} \frac{\partial u_j}{\partial y_m} \frac{\partial u_i}{\partial y_j} dV_y \\ + \frac{1}{2} \iiint_{\mathcal{D}} p_{ji} p_{mk} \tau_{ik} \frac{\partial u_n}{\partial y_j} \frac{\partial u_n}{\partial y_m} dV_y . \end{aligned} \tag{3.7}$$

Now, we have, from (3.2) and (3.5), that

$$
\begin{aligned}
\left| \iiint_{D} p_{mk} \tau_{ik} \frac{\partial u_j}{\partial y_m} \frac{\partial u_i}{\partial y_j} dV_y \right| &\le \iiint_{D} |p_{mk}| |\tau_{ik}| \left|\frac{\partial u_j}{\partial y_m}\right| \left|\frac{\partial u_i}{\partial y_j}\right| dV_y \\
&\le C_2 \varepsilon \left(\frac{d}{D}\right)^3 \iiint_{D} \frac{\partial u_i}{\partial y_k} \frac{\partial u_i}{\partial y_k} dV_y \qquad (3.8) \\
&\le C_3 \varepsilon \iiint_{D} \tau_{ik} e_{ik} dV_y .
\end{aligned}
$$

Similarly,

$$
\left| \iiint_{D} p_{ji} p_{mk} \tau_{ik} \frac{\partial u_n}{\partial y_j} \frac{\partial u_n}{\partial y_m} dV_y \right| \le C_4 \varepsilon \iiint_{D} \tau_{ik} e_{ik} dV_y . \qquad (3.9)
$$

Thus, for sufficiently small ε, the sum of the last two terms on the right side of (3.7) is less than $\frac{1}{2} \iiint_{D} \tau_{ik} e_{ik} dV_y$ and therefore

$$
\frac{1}{2} \iiint_{D} \tau_{ik} e_{ik} dV_y \le \iiint_{D} F_i u_i dV_y + \iint_{\partial D} T_i u_i dS_y . \qquad (3.10)
$$

Since the body is in equilibrium, the surface tractions and body forces satisfy the consistency relations (see [15], pp. 127-128)

$$
\iint_{\partial D} T_i dS_y + \iiint_{D} F_i dV_y = 0 , \qquad (3.11a)
$$

$$
\iint_{\partial D} (y_i T_k - y_k T_i) dS_y + \iiint_{D} (y_i F_k - y_k F_i) dV_y = 0 . \qquad (3.11b)
$$

The first equation above expresses the vanishing of the resultant force, while the second expresses the vanishing of the resulting moment in the deformed body.

We now define

$$
v_i = u_i + \alpha_{ij} y_j + \beta_i \qquad (3.12)
$$

where α_{ij} is a real constant anti-symmetric tensor $(\alpha_{ik} = -\alpha_{ki})$ and β_i is a constant vector, both of which will be chosen later.

From the relations (3.11) we see that

$$
\iiint_{D} F_i u_i dV_y + \iint_{\partial D} T_i u_i dS_y = \iiint_{D} F_i v_i dV_y + \iint_{\partial D} T_i v_i dS_y , \qquad (3.13)
$$

and then, from the Cauchy-Schwarz inequality, we obtain

$$\iiint_{\mathcal{D}} F_i u_i dV_y + \iint_{\partial\mathcal{D}} T_i u_i dS_y \le [\iiint_{\mathcal{D}} F_i F_i dV_y]^{\frac{1}{2}} [\iiint_{\mathcal{D}} v_i v_i dV_y]^{\frac{1}{2}} \tag{3.14}$$

$$+ [\iint_{\partial\mathcal{D}} T_i T_i dS_y]^{\frac{1}{2}} [\iint_{\partial\mathcal{D}} v_i v_i dS_y]^{\frac{1}{2}} .$$

Combination of (3.14) and (3.10) yields

$$\iiint_{\mathcal{D}} \tau_{ik} e_{ik} dV_y \le 2[\iiint_{\mathcal{D}} F_i F_i dV_y]^{\frac{1}{2}} [\iiint_{\mathcal{D}} v_i v_i dV_y]^{\frac{1}{2}} \tag{3.15}$$

$$+ 2 [\iint_{\partial\mathcal{D}} T_i T_i dS_y]^{\frac{1}{2}} [\iint_{\partial\mathcal{D}} v_i v_i dS_y]^{\frac{1}{2}} .$$

We now recall from the statement of the theorem that $y_i n_i/\sqrt{y_k y_k} \ge -\frac{1}{4}$ on $\partial\mathcal{D}$ and that the interior of the sphere $\Sigma : y_i y_i = \frac{1}{16} d^2$ is contained in $\mathcal{D}$. Next we refer to the work of Bramble and Payne [1] who proved that if the coefficients α_{ik} and β_i are chosen such that

$$\iint_{\Sigma} v_i dS_y = \iint_{\Sigma} (v_i y_j - v_j y_i) dS_y = 0 , \tag{3.16}$$

then, for a sufficiently large universal constant q_1,

$$\iint_{\partial\mathcal{D}} v_i v_i dS_y \le C_5 d \left(\frac{D}{d}\right)^{q_1} \iiint_{\mathcal{D}} \tilde{W}(\theta) dV_y , \tag{3.17a}$$

and

$$\iiint_{\mathcal{D}} v_i v_i dV_y \le C_6 d^2 \left(\frac{D}{d}\right)^{q_1+1} \iiint_{\mathcal{D}} \tilde{W}(\theta) dV_y , \tag{3.17b}$$

where

$$\theta_{ik} = \frac{1}{2} \left(\frac{\partial v_i}{\partial y_k} + \frac{\partial v_k}{\partial y_i}\right) \tag{3.18a}$$

and

$$\tilde{W}(\theta) = \mu\theta_{ik}\theta_{ik} + \frac{1}{2} \lambda\theta_{ii}\theta_{kk} . \tag{3.18b}$$

The quantities θ_{ik} and $\tilde{W}(\theta)$ represent, respectively, the linearized strain and the corresponding linearized energy density function. We note that

$$\tilde{W}(\theta) \le (\mu + \frac{3}{2} \lambda)\theta_{ik}\theta_{ik} . \tag{3.19}$$

From (3.12), we see that

$$\theta_{ik} = \frac{1}{2} \left(\frac{\partial u_i}{\partial y_k} + \frac{\partial u_k}{\partial y_i}\right) , \tag{3.20}$$

and then, from (3.20) and (2.10),

$$\theta_{ik} = e_{ik} - \frac{1}{2} p_{mk} \frac{\partial u_j}{\partial y_m} \frac{\partial u_i}{\partial y_j} - \frac{1}{2} p_{mi} \frac{\partial u_j}{\partial y_m} \frac{\partial u_k}{\partial y_j} - \frac{1}{2} p_{ji} p_{mk} \frac{\partial u_n}{\partial y_j} \frac{\partial u_n}{\partial y_m} . \qquad (3.21)$$

The elements of p_{ij} and $\frac{\partial u_i}{\partial y_j}$ are uniformly bounded and the tensor $\frac{\partial u_i}{\partial y_k}$ satisfies (3.5). Therefore,

$$\iiint_{\mathcal{D}} \theta_{ik}\theta_{ik} dV_y \leq C_7 \frac{D^3}{d^3} \iiint_{\mathcal{D}} \tau_{ik} e_{ik} dV_y , \qquad (3.22)$$

and, from (3.19),

$$\iiint_{\mathcal{D}} \tilde{W}(\theta) dV_y \leq C_8 \frac{D^3}{d^3} \iiint_{\mathcal{D}} \tau_{ik} e_{ik} dV_y . \qquad (3.23)$$

With $q_2 = q_1 + 3$, Equations (3.23) and (3.17) imply that

$$\iint_{\partial\mathcal{D}} v_i v_i dS_y \leq C_9 d \left(\frac{D}{d}\right)^{q_2} \iiint_{\mathcal{D}} \tau_{ik} e_{ik} dV_y , \qquad (3.24a)$$

and

$$\iiint_{\mathcal{D}} v_i v_i dV_y \leq C_{10} d^2 \left(\frac{D}{d}\right)^{q_2+1} \iiint_{\mathcal{D}} \tau_{ik} e_{ik} dV_y . \qquad (3.24b)$$

Combining (3.24) with (3.15), we obtain

$$\sqrt{\iiint_{\mathcal{D}} \tau_{ik} e_{ik} dV_y} \leq 2\sqrt{C_{10}}\, d \left(\frac{D}{2}\right)^{\frac{q_2+1}{2}} \|F\| + 2\sqrt{C_9 d} \left(\frac{D}{d}\right)^{\frac{q_2}{2}} \|T\| . \qquad (3.25)$$

Finally, the remarks in the paragraph preceding (3.2) imply that there exists a constant C_{11} such that

$$U(\mathcal{D}) \leq C_{11} \iiint_{\mathcal{D}} \tau_{ik} e_{ik} dV_y , \qquad (3.26)$$

and then (3.25) and (3.26) lead to the desired result, Equation (3.1).

4. SOME ADDITIONAL REMARKS.

As stated earlier in Section 1, the techniques used here are also applicable to non-isotropic and non-homogeneous bodies and the restriction here to a homogeneous isotropic body was only for the purpose of simplifying the presentation. The convexity assumption on $\mathcal{D}_R$ can also be relaxed somewhat; the important thing is that the domain be such that it is possible to obtain an inequality of the form (3.3) by an appropriate rotation. (See [6], [7] for a discussion of the factors involved).

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For the traction boundary value problem in nonlinear elastostatics for a body which is convex in its undeformed reference state and with the assumption of sufficiently small strains (but not necessarily small displacement gradients), an upper bound is obtained for the elastic strain energy in terms of the L_2-integral norms of the surface tractions and body forces with the constant depending only upon the ratio of the outer and inner diameters and the physical constants of the material.

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ABSTRACT (continued)

This result extends previous known results in linear elasticity (infinitesimal displacement gradients) and finite elasticity (small but finite displacement gradients) into the small strain theory of nonlinear elasticity.